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GEOMETRIC PROPERTIES OF THE SECTIONS OF SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

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ABSTRACT. In this paper we establish several geometric properties of the cross sections of generalized solutions ϕ to the Monge-Ampère equation $\det D^2\phi = \mu$, when the measure μ satisfies a doubling property. A main result is a characterization of the doubling measures μ in terms of a geometric property of the cross sections of ϕ. This is used to obtain estimates of the shape and invariance properties of the cross sections that are valid under appropriate normalizations.

0. INTRODUCTION

Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a convex function. A supporting hyperplane to ϕ at the point $(x_0, \phi(x_0))$ is an affine function $\ell(x) = \phi(x_0) + p \cdot (x - x_0)$ such that $\phi(x) \geq \ell(x)$ for all $x \in \mathbf{R}^n$. Given $t > 0$, a section of ϕ at height t is the convex set

$$S_\phi(x_0, p, t) = \{x \in \mathbf{R}^n : \phi(x) < \ell(x) + t\}.$$

If ϕ is smooth, then ℓ is unique, $\ell(x) = \phi(x_0) + \nabla\phi(x_0) \cdot (x - x_0)$, and we write

$$S_\phi(x_0, t) = S_\phi(x_0, p, t).$$

The normal mapping of ϕ is the set-valued function $\nabla\phi : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ defined by

$$\nabla\phi(x_0) = \{p : \phi(x) \geq \phi(x_0) + p \cdot (x - x_0),\ \forall x \in \mathbf{R}^n\}.$$

If $E \subset \mathbf{R}^n$, then $\nabla\phi(E) = \bigcup_{x\in E} \nabla\phi(x)$. By a classical theorem of Aleksandrov, the class of sets E such that $\nabla\phi(E)$ is Lebesgue measurable is a σ-algebra that contains the Borel sets and one can define the Monge-Ampère measure associated with ϕ as the Borel measure μ given by $\mu(E) = |\nabla\phi(E)|$; see [Ch-Y].

The purpose in this paper to analyze in detail several important geometric properties of the sections of the convex function ϕ when its associated Monge-Ampère measure μ satisfies a doubling condition. The interest in these properties comes from the study of the solutions of the Monge-Ampère equation, and its linearizations, both elliptic and parabolic, and from real harmonic analysis; see [C1], [C2], [C-G1], [C-G2] and [H]. Some of these properties have been used in those references to prove a lemma of Besicovitch's type and a Calderón-Zygmund decomposition in terms of sections which allows us to establish estimates of the solutions to the linearized Monge-Ampère equation. Also, some of these properties imply that $\mathbf{R}^n$

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equipped with the Monge-Ampère measure and the family of sections is a space of homogenous type; see [A-F-T].

One of the main results in this paper is a geometric characterization of Monge-Ampère doubling measures; see Theorem 2.1 below. This result is used to obtain estimates of the shape of the sections and invariance properties valid under appropriate normalizations. In this framework, we obtain another proof of a result due to Caffarelli [C1] about the strict convexity of solutions to the Monge-Ampère equation.

We shall assume throughout this paper that the sections $S_\phi(x_0, p, t)$ are bounded sets. Let x_0^* be the center of mass of $S_\phi(x_0, p, t)$. If $\lambda > 0$, then $\lambda\, S_\phi(x_0, p, t)$ denotes the λ-dilation of $S_\phi(x_0, p, t)$ with respect to its center of mass; that is

$$\lambda\, S_\phi(x_0, p, t) = \{x_0^* + \lambda(x - x_0^*) : x \in S_\phi(x_0, p, t)\}.$$

We introduce the following two doubling conditions. We say that the Borel measure ν is doubling with respect to the center of mass on the sections of ϕ if there exist constants $C > 0$ and $0 < \alpha < 1$ such that for all sections $S_\phi(x, p, t)$,

$$\text{(DC)} \qquad \nu\left(S_\phi(x, p, t)\right) \le C\, \nu\left(\alpha\, S_\phi(x, p, t)\right).$$

On the other hand, we say that ν is doubling with respect to the parameter on the sections of ϕ if there exists a constant $C' > 0$ such that for all sections $S_\phi(x, p, t)$,

$$\text{(DP)} \qquad \nu\left(S_\phi(x, p, t)\right) \le C'\, \nu\left(S_\phi(x, p, t/2)\right).$$

Some comments about these doubling conditions are as follows. If the Monge-Ampère measure μ associated with the strictly convex function ϕ satisfies (DC), then it was shown in [C2] that ϕ is regular. Condition (DP) appears in [C-G1] to show the covering lemma of the Calderón-Zygmund type mentioned before. It will be shown that condition (DC) implies (DP), but the converse is in general false; see Corollary 2.1 and the subsequent remark. For examples of measures satisfying (DC) see Remark 2.2.

The assumption made that the sections $S_\phi(x, p, t)$ are bounded sets allows ϕ to have finite segments of lines in the graph. It is easy to see that if ϕ is strictly convex, then all sections $S_\phi(x, p, t)$ are bounded sets; otherwise, the graph of ϕ may contain half-lines. As a consequence of Theorem 2.3, it follows that if the sections of ϕ are bounded sets and (DC) holds, then ϕ is strictly convex; see Remark 2.3.

We remark that if the convex function ϕ is defined only on a convex open set $\Omega \subset \mathbf{R}^n$, then the results of this paper hold true with straightforward modifications if we add to the hypothesis that the sections are bounded the following: given $x \in \Omega$ there exists t_0 such that $\overline{S_\phi(x, p, t)} \subset \Omega$ for all $t \le t_0$ and $p \in \nabla\phi(x)$.

This paper is organized as follows. In section 1, we present the basic facts needed to prove the properties of the cross sections. Section 2 is subdivided into four parts. The characterization of Monge-Ampère doubling measures in contained in 2.1. Section 2.2 contains the proof of the engulfing property. A quantitative estimate of the size of the cross sections and some consequences are contained in 2.3. Finally, in 2.4 we obtain Caffarelli's strict convexity result.

1. Preliminary results

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible affine transformation, i.e., $Tx = Ax + b$ where A is an $n \times n$ invertible real matrix and $b \in \mathbf{R}^n$. If $\phi : \mathbf{R}^n \to \mathbf{R}$ and $\lambda > 0$, let $\psi_\lambda(y) = \dfrac{1}{\lambda}\phi(T^{-1}y)$. The function $\ell(x) = \phi(x_0) + p \cdot (x - x_0)$ is a supporting hyperplane of ϕ at the point $(x_0, \phi(x_0))$ if and only if $\bar{\ell}(y) = \psi_\lambda(Tx_0) + \frac{1}{\lambda}(A^{-1})^t p \cdot (y - Tx_0)$ is a supporting hyperplane of the function ψ_λ at the point $(Tx_0, \psi_\lambda(Tx_0))$. Let μ and $\bar{\mu}$ be the Monge-Ampère measures associated with ϕ and ψ_λ respectively; that is

$$\mu(E) = |\nabla\phi(E)|, \qquad \bar{\mu}(E) = |\nabla\psi_\lambda(E)|.$$

Note that

$$\frac{1}{\lambda}\left(A^{-1}\right)^t (\nabla\phi(E)) = \nabla\psi_\lambda\left(TE\right),$$

and consequently

$$\bar{\mu}\left(TE\right) = \frac{1}{\lambda^n}|\det T^{-1}|\mu(E).$$

In addition, the sections of ϕ and ψ_λ are related by the following formula:

$$T(S_\phi(x,p,t)) = S_{\psi_\lambda}(Tx, \frac{1}{\lambda}(A^{-1})^t p, \frac{t}{\lambda}). \tag{1-0}$$

Hence, noting that $T\left(\alpha S_\phi(x,p,t)\right) = \alpha T\left(S_\phi(x,p,t)\right)$, it follows that if μ satisfies either (DC) or (DP) on the sections of ϕ, then the measure $\bar{\mu}$ satisfies (DC) or (DP) on the sections of ψ_λ respectively and with the same constants.

All properties of the sections will follow from the following two basic facts. The first one is the following variant of a geometric lemma due to Fritz John [C1].

Fritz John's Lemma. *Let S be a bounded and convex set in $\mathbf{R}^n$ with non-empty interior. Consider all the ellipsoids that contain S and that are centered at the center of mass of S. Let E be the ellipsoid of minimum volume. Then there exists a constant $\alpha_n > 0$ depending only on n such that*

$$\alpha_n E \subset S \subset E,$$

where $\alpha_n E$ means the α_n-dilation of E with respect to its center.

Since E is an ellipsoid, there is an affine transformation T such that $T(E) = B(0,1)$. Then

$$B(0,\alpha_n) \subset T(S) \subset B(0,1). \tag{1-1}$$

Here $B(x,t)$ denotes the Euclidean ball with center x and radius t. The set $T(S)$ shall be called a *normalization* of S, and T shall be called an affine transformation that *normalizes* S. The center of mass of $T(S)$ is 0 and by taking Lebesgue measure in (1-1) it follows that

$$\alpha_n^n\,\Omega_n \frac{1}{|S|} \le |\det T| \le \Omega_n \frac{1}{|S|}, \tag{1-2}$$

where Ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. We say that the convex set S is normalized when its center of mass is 0 and $B(0,\alpha_n) \subset S \subset B(0,1)$. If S is a section for the function ϕ, then by (1-0) any normalization of S is also a section corresponding to the function ψ_λ.

The second basic fact needed is the following estimate due to A. D. Aleksandrov [A]; see also [R-T].

Aleksandrov's estimate. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded and convex set, and $u \in C\left(\bar{\Omega}\right)$, u convex and $u|_{\partial\Omega} = 0$. Then there exists a constant c_n depending only on n such that*

$$|u(x)|^n \le c_n \left(\operatorname{diam}(\Omega)\right)^{n-1} \operatorname{d}(x, \partial\Omega)\, \mu(\Omega), \qquad \text{for all } x \in \Omega,$$

where $\mu(\Omega) = |\nabla u(\Omega)|$ and ∇u denotes the normal mapping of u.

The following lemmas give estimates of the size of the slopes of supporting hyperplanes to a convex function; see [C1] and [C-G2].

Lemma 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex open set and ϕ a convex function in Ω such that $\phi \le 0$ on $\partial\Omega$. If $x \in \Omega$ and $\ell(y) = \phi(x) + p \cdot (y - x)$ is a supporting hyperplane to ϕ at the point $(x, \phi(x))$, then*

$$|p| \le \frac{-\phi(x)}{\operatorname{d}(x, \partial\Omega)}. \tag{1-3}$$

More generally, if $\overline{\Omega'} \subset \Omega$, then

$$\nabla\phi\left(\Omega'\right) \subset B\left(0, \frac{\max_{\Omega'}(-\phi)}{\operatorname{d}(\Omega', \partial\Omega)}\right). \tag{1-4}$$

Proof. Assume $p \ne 0$. We have $\phi(y) \ge \phi(x) + p \cdot (y - x)$, for all $y \in \Omega$. If $0 < r < d(x, \partial\Omega)$, then $y_0 = x + r\dfrac{p}{|p|} \in \Omega$ and $0 \ge \phi(y_0) \ge \phi(x) + r|p|$, which proves the lemma. □

Lemma 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex open set, let ϕ be a convex function in Ω and let $\phi = 0$ on $\partial\Omega$. Given $\lambda > 0$, let $\lambda\Omega$ denote the set obtained by λ-dilating Ω with respect to its center of mass. There exists a positive constant C_n depending only on n such that*

$$B\left(0, \frac{1}{2}\frac{|\min_\Omega \phi|}{\operatorname{diam}(\Omega)}\right) \subset \nabla\phi\left(\lambda\Omega\right),$$

for $\lambda_n < \lambda < 1$ where

$$\lambda_n = \max\left\{\frac{1}{2}, 1 - C_n \left(\frac{|\min_\Omega \phi|}{\operatorname{diam}(\Omega)}\right)^n \frac{1}{\mu(\Omega)}\right\}.$$

Proof. Let x_0 be the center of mass of Ω, and $\lambda\Omega = \{x_0 + \lambda(x - x_0) : x \in \Omega\}$. If $x \in \partial\left(\lambda\Omega\right)$, then $d(x, \partial\Omega) \le (1 - \lambda)\operatorname{diam}(\Omega)$. Thus, by the Aleksandrov estimate

$$|\phi(x)|^n \le c_n \left(\operatorname{diam}(\Omega)\right)^n \left(1 - \lambda\right) \mu(\Omega)$$

for $x \in \partial(\lambda\Omega)$. Let us pick $\lambda > 1/2$ such that

$$c_n \left(\operatorname{diam}(\Omega)\right)^n \left(1 - \lambda\right) \mu(\Omega) \le \left(\frac{1}{2}|\min_\Omega \phi|\right)^n.$$

Let $m_\lambda = \min_{\partial(\lambda\Omega)} \phi(x)$, $m = \min_\Omega \phi = \phi(z_0)$, and consider the cone Γ_λ in $\mathbf{R}^{n+1}$ passing through $\{(x, \phi(x)) : x \in \partial(\lambda\Omega)\}$ and with vertex at (z_0, m). If χ_{Γ_λ} denotes the normal mapping corresponding to the cone Γ_λ, then $\chi_{\Gamma_\lambda}\left(\lambda\Omega\right) \subset \nabla\phi\left(\lambda\Omega\right)$. Since $m_\lambda > -\dfrac{1}{2}|\min_\Omega \phi|$, and noticing that $B(0, \frac{m_\lambda - m}{\operatorname{diam}(\lambda\Omega)}) \subset \chi_{\Gamma_\lambda}(\lambda\Omega)$, we get

$$B\left(0, \frac{1}{2}\frac{|\min_\Omega \phi|}{\operatorname{diam}(\lambda\Omega)}\right) \subset \chi_{\Gamma_\lambda}\left(\lambda\Omega\right).$$

This yields the inequality of the lemma. □

By combining Lemmas 1.1 and 1.2, we obtain the following proposition; see [C1] and Lemma 1.1 of [C-G2].

Proposition 1.1. *Let Ω be a convex domain in $\mathbf{R}^n$ with center of mass equal* 0, $B(0,\alpha_n) \subset \Omega \subset B(0,1)$*, and ϕ a convex function in Ω, $\phi = 0$ on $\partial\Omega$. Let μ be the Monge-Ampère measure associated with ϕ and assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that $\mu(\Omega) \leq C\,\mu(\alpha\,\Omega)$. Then*

$$C_1 |\min_{\Omega} \phi|^n \leq \mu(\Omega) \leq C_2 |\min_{\Omega} \phi|^n$$

where C_1, C_2 are positive constants depending only on C, α and n.

2. Properties of the sections

2.1. The Monge-Ampère measures satisfying (DC)**.** The purpose of this section is to give a geometric characterization of the Monge-Ampère measures satisfying (DC) and also to compare (DC) and (DP).

We begin with the following lemma.

Lemma 2.1. *Let $0 < \lambda < 1$. Then*

$$\lambda\, S_\phi(x_0, p, t) \subset S_\phi(x_0, p, \Big(1 - (1-\lambda)\frac{\alpha_n}{2}\Big)\, t).$$

Proof. Let x_0^* be the center of mass of $S_\phi(x_0,p,t)$, $\ell(x) = \phi(x_0) + p\cdot(x - x_0)$, the hyperplane defining $S_\phi(x_0,p,t)$, and let T be an affine transformation that normalizes $S_\phi(x_0,p,t)$. We have $T\,(\lambda\, S_\phi(x_0,p,t)) = \lambda\, T\,(S_\phi(x_0,p,t))$ and if $\psi(y) = \phi(T^{-1}y)$ and $q = (T^{-1})^t p$, then by (1-0) $T\,(\lambda\, S_\phi(x_0,p,t)) = \lambda\, S_\psi(Tx_0,q,t)$. We claim that

$$\lambda\, S_\psi(Tx_0, q, t) \subset S_\psi(Tx_0, q, \Big(1 - (1-\lambda)\frac{\alpha_n}{2}\Big)\, t). \tag{2-1}$$

In fact, since $0 \in S_\psi(Tx_0,q,t)$ and by the convexity, there exists a point $\xi \in \partial S_\psi(Tx_0,q,t)$ and $0 < \theta \leq 1$ such that $0 = \theta T x_0 + (1-\theta)\xi$. Hence, by (1-1)

$$\theta \geq \frac{|\xi|}{|Tx_0 - \xi|} \geq \frac{\alpha_n}{2}.$$

Let $\bar\ell(x) = \psi(Tx_0) + q\cdot(x - Tx_0)$ be the hyperplane defining $S_\psi(Tx_0,q,t)$. Since $\psi(y) - \bar\ell(y)$ is convex, it follows that

$$\psi(0) - \bar\ell(0) \leq \theta\,\big(\psi(Tx_0) - \bar\ell(Tx_0)\big) + (1-\theta)\,\big(\psi(\xi) - \bar\ell(\xi)\big) \leq (1 - \frac{\alpha_n}{2})t.$$

Hence, if $y \in S_\psi(Tx_0,q,t)$, then

$$\begin{aligned}\psi(\lambda y) - \bar\ell(\lambda y) &\leq \lambda\,\big(\psi(y) - \bar\ell(y)\big) + (1-\lambda)\,\big(\psi(0) - \bar\ell(0)\big)\\ &\leq \lambda t + (1-\lambda)(1 - \frac{\alpha_n}{2})t\\ &= \Big(1 - (1-\lambda)\frac{\alpha_n}{2}\Big)\,t,\end{aligned}$$

and the claim follows. Lemma 2.1 follows by taking T^{-1} in (2-1). □

Corollary 2.1 ((DC) implies (DP))**.** *Let μ be a measure satisfying* (DC)*. Then μ satisfies* (DP).

Proof. By Lemma 2.1

$$\mu(S_\phi(x,p,t)) \le C\,\mu(\alpha S_\phi(x,p,t)) \le C\,\mu(S_\phi(x,p,\theta t)), \qquad \theta = 1-(1-\alpha)\frac{\alpha_n}{2}.$$

By successive application of the last inequality

$$\mu(S_\phi(x,p,t)) \le C^k\,\mu(S_\phi(x,p,\theta^k t)),$$

and by taking k such that $\theta^k \le 1/2$, the corollary follows. □

Remark 2.1. The converse of Corollary 2.1 is false. The function $\phi(x) = e^x$, $x \in \mathbf{R}$, is strictly convex and the corresponding Monge-Ampère measure satisfies (DP) but not (DC). For such ϕ we observe that

(1) $S_\phi(x_0,t) = x_0 + S_\phi(0, te^{-x_0})$,
(2) $\mu\left(S_\phi(x_0,t)\right) = e^{x_0}|S_\phi(0,te^{-x_0})| = e^{x_0}\mu\left(S_\phi(0,te^{-x_0})\right)$.

Any interval of the form $(0,L)$, $L>0$, is a section at some point. Given $0<\alpha<1$, $\alpha\,(0,L) = \dfrac{L}{2} + \alpha\left((0,L) - \dfrac{L}{2}\right) = \left(\dfrac{1-\alpha}{2}L, \dfrac{1+\alpha}{2}L\right)$. Thus

$$\frac{\mu\left((0,L)\right)}{\mu\left(\alpha\,(0,L)\right)} = \frac{e^L - 1}{e^{(1+\alpha)L/2} - e^{(1-\alpha)L/2}} \to \infty,$$

if $L \to \infty$ and hence (DC) fails for any $0<\alpha<1$.

Let us show that (DP) holds for this measure. We have that $S_\phi(0,t) = (m_t, M_t)$, where $m_t < 0 < M_t$, and the following estimates:

(3) $\ln t \le M_t \le \ln 2t$ and $-2t \le m_t \le -t$ for $t \ge 5$.
(4) There exist positive constants ϵ and c such that $\epsilon\sqrt{t} \le M_t \le \sqrt{2t}$ and $-c\sqrt{t} \le m_t \le -\sqrt{2t}$ for $0 < t \le 5$.

We write

$$\frac{\mu\left(S_\phi(0,t)\right)}{\mu\left(S_\phi(0,t/2)\right)} = \frac{|S_\phi(0,t)|}{|S_\phi(0,t/2)|} = \frac{M_t - m_t}{M_{t/2} - m_{t/2}} = (*),$$

and from (3) and (4) it follows that $(*) \le C$ for all $t>0$. This combined with (1) and (2) yields

$$\frac{\mu\left(S_\phi(x_0,t)\right)}{\mu\left(S_\phi(x_0,t/2)\right)} = \frac{e^{x_0}|S_\phi(0,te^{-x_0})|}{e^{x_0}|S_\phi(0,t/2e^{-x_0})|} \le C'. \quad \square$$

Remark 2.2. If $p(x)$ is a positive polynomial in $\mathbf{R}^n$, then we shall show that the measure $p(x)\,dx$ satisfies (DC) on the bounded sections of any convex function ϕ and with a constant depending only on the degree of p. We use the following result of Ricci and Stein: let $p(x)$ be a polynomial of degree d in $\mathbf{R}^n$, then

$$\int_{B_1(0)} |p(x)|^{-\mu}\,dx \le C_{\mu,d}\left(\int_{B_1(0)} |p(x)|\,dx\right)^{-\mu},$$

where $\mu d < 1$ and the constant $C_{\mu,d}$ is independent of p; see [S], p. 219. Hence, to show our claim, let $p(x) > 0$ have degree d and take $r > 1$ and $\gamma > 0$ exponents that will be chosen in a moment. By Hölder's inequality

$$\int_{B_1(0)} dx \le C_{\gamma,r,d}\left(\int_{B_1(0)} p(x)^{2\gamma r}\,dx\right)^{1/r}\left(\int_{B_1(0)} p(x)^2\,dx\right)^{-\gamma}$$

for $\gamma \frac{r}{r-1} 2d < 1$. If $\gamma r = 1/2$, then $r > d+1$ and we obtain

$$\int_{B_1(0)} p(x)^2\,dx \le C_d \left(\int_{B_1(0)} p(x)\,dx \right)^2,$$

for every positive polynomial p of degree d. Hence by Schwartz's inequality

$$\begin{aligned} \int_{B_1(0)\setminus B_{1-\epsilon}(0)} p(x)\,dx &\le \omega_n (1-(1-\epsilon)^n)^{1/2} C_d \int_{B_1(0)} p(x)\,dx \\ &\le C\epsilon^{1/2} \int_{B_1(0)} p(x)\,dx, \end{aligned}$$

with $C = C(n,d)$. Thus

$$\int_{B_1(0)} p(x)\,dx \le 2 \int_{B_{1-\epsilon}(0)} p(x)\,dx$$

for ϵ sufficiently small. By changing variables

$$\int_{B_r(0)} p(x)\,dx \le 2 \int_{B_{(1-\epsilon)r}(0)} p(x)\,dx$$

for all $r > 0$, and by iteration it follows that

$$\int_{B_1(0)} p(x)\,dx \le C \int_{B_{\alpha_n/2}(0)} p(x)\,dx,$$

where α_n is the constant in the variant of the Fritz John lemma.

Now, let S be a section of ϕ and T an affine transformation that normalizes S, i.e., $B_{\alpha_n}(0) \subset S^* = T(S) \subset B_1(0)$. We then have

$$\begin{aligned} \int_S p(x)\,dx &= \int_{S^*} p(T^{-1}y) |\det T^{-1}|\,dy \le |\det T^{-1}| \int_{B_1(0)} p(T^{-1}y)\,dy \\ &\le C|\det T^{-1}| \int_{B_{\alpha_n/2}(0)} p(T^{-1}y)\,dy \le C|\det T^{-1}| \int_{\frac{1}{2}T(S)} p(T^{-1}y)\,dy \\ &= C \int_{\frac{1}{2}S} p(x)\,dx, \end{aligned}$$

since $T(\frac{1}{2}S) = \frac{1}{2}T(S)$. This completes the remark. □

The following theorem is one of the main results in this paper and gives a geometric characterization of doubling Monge-Ampère measures.

Theorem 2.1. *Let μ be the Monge-Ampère measure associated with the convex function ϕ. The following statements are equivalent:*

(i) *μ satisfies the doubling condition* (DC).

(ii) *There exist $0 < \tau,\ \lambda < 1$ such that for all $x_0 \in \mathbf{R}^n$ and $t > 0$,*

$$S_\phi(x_0, p, \tau t) \subset \lambda\, S_\phi(x_0, p, t). \tag{2-2}$$

Proof. Let us assume (i). We shall show that there exists a dimensional constant $0 < \beta_n \le 1$ such that (2-2) holds for all τ and λ such that $0 < \tau < 1$ and $1 - \beta_n(1-\tau)^n \le \lambda < 1$. Let T be an affine transformation that normalizes $S_\phi(x_0, p, t)$, x_0^* is the center of mass, and $\psi(y) = \phi(T^{-1}y)$. Let $0 < \lambda < 1$. By (1-0) we have

$T(S_\phi(x_0,p,\lambda t)) = S_\psi(Tx_0,q,\lambda t)$, where $q=(T^{-1})^t p$. Since the center of mass of $S_\psi(Tx_0,q,\lambda t)$ is $Tx_0^*=0$, we have

$$T(\lambda\, S_\phi(x_0,p,t)) = T\{x_0^*+\lambda(x-x_0^*) : x\in S_\phi(x_0,p,t)\}$$
$$= \{\lambda Tx : x\in S_\phi(x_0,p,t)\} = \lambda S_\psi(Tx_0,q,t).$$

Set $\psi^*(y)=\psi(y)-\psi(Tx_0)-q\cdot(y-Tx_0)-t$. Then $\nabla\psi^*=\nabla\psi-q$ and $\psi^*|_{\partial S_\psi(Tx_0,q,t)} = 0$.

If $y\in S_\psi(Tx_0,q,t)\setminus\lambda\, S_\psi(Tx_0,q,t)$, then $d(y,\partial S_\psi(Tx_0,q,t))\le 1-\lambda$, and by Aleksandrov's estimate and Proposition 1.1

$$|\psi^*(y)|^n \le c_n d(y,\partial S_\psi(Tx_0,q,t))|\min_{S_\psi(Tx_0,q,t)}\psi^*(y)|^n \le c_n(1-\lambda)t^n.$$

Hence, $\psi^*(y)\ge -c_n^{-1/n}(1-\lambda)^{1/n}t$ which implies

$$\psi(y)-\psi(Tx_0)-q\cdot(y-Tx_0)\ge\left(1-c_n^{-1/n}(1-\lambda)^{1/n}\right)t\ge\tau t;$$

that is, $y\notin S_\psi(Tx_0,q,\tau t)$. Therefore

$$S_\psi(Tx_0,q,\tau t)\subset\lambda\, S_\psi(Tx_0,q,t).$$

Hence, (2-2) follows by taking T^{-1}.

Now we prove that (ii) implies (i). Let T be an affine transformation normalizing $S_\phi(x_0,p,t)$ and let $\psi(y)=(\phi(T^{-1}y)-\phi(x_0)-q\cdot(y-Tx_0))/t$. Obviously, by (2-2) we have

$$S_\psi(Tx_0,0,\tau)\subset\lambda S_\psi(Tx_0,0,1).$$

If $x\in S_\psi(Tx_0,0,\tau)$ and q^* is the slope of a supporting hyperplane to ψ at $(x,\psi(x))$, then by Lemma 1.2,

$$|q^*|\le\frac{1}{\mathrm{d}(x,\partial S_\psi(Tx_0,0,1))}\le C_{n,\lambda},$$

and hence

$$\bar\mu\left(S_\psi(Tx_0,0,\tau)\right)\le C'_{n,\lambda}. \tag{2-3}$$

On the other hand, by applying Aleksandrov's estimate to $\psi(y)-\frac{\tau}{2}$ in $S_\psi(Tx_0,0,\frac{\tau}{2})$ it follows that

$$\left(\frac{\tau}{2}\right)^n\le C_n\bar\mu\left(S_\psi(Tx_0,0,\frac{\tau}{2})\right). \tag{2-4}$$

From (2-3) and (2-4) we obtain

$$\bar\mu\left(S_\psi(Tx_0,0,\tau)\right)\le C\bar\mu\left(S_\psi(Tx_0,0,\frac{\tau}{2})\right),$$

which implies

$$\mu\left(S_\phi(x_0,p,\tau t)\right)\le C\mu\left(S_\phi(x_0,p,\frac{\tau}{2}t)\right),$$

where the constant C is independent of t. If we pick k such that $2^{-k}<\tau$, then by iteration we obtain

$$\mu\left(S_\phi(x_0,p,t)\right)\le C\mu\left(S_\phi(x_0,p,\frac{t}{2})\right)\le C^k\mu\left(S_\phi(x_0,p,2^{-k}t)\right)\le C'\mu\left(S_\phi(x_0,p,\tau t)\right).$$

Now (i) follows from (2-2). □

As a first consequence of our characterization we obtain the following corollary.

Corollary 2.2. *Let μ be the Monge-Ampère measure associated with the convex function ϕ and assume that μ satisfies* (DC). *Let T be an affine transformation that normalizes the section $S_\phi(x,p,t)$, (in particular, by* (1-0), $T(S_\phi(x,p,t)) = S_\psi(Tx,q,t)$ *where $\psi(y)=\phi(T^{-1}y)$, and $q=(T^{-1})^t p$). Then*

(i) *there exists $c_0>0$ depending only on the constant in* (DC) *such that*

$$\mathrm{d}\left(S_\psi(Tx,q,\tau t), \partial S_\psi(Tx,q,t)\right) \geq c_0(1-\tau)^n \qquad \textit{for all } 0<\tau<1;$$

(ii) *there exists $C>0$ depending only on the constant in* (DC) *and n such that if $y \notin S_\phi(x,p,t)$, then*

$$B(T(y), C\epsilon^n) \cap T\left(S_\phi(x,p,(1-\epsilon)t)\right) = \emptyset \qquad \textit{for all } 0<\epsilon<1.$$

Proof. (i) By Theorem 2.1, $S_\psi(Tx,q,\tau t) \subset \lambda S_\psi(Tx,q,t)$ with $\lambda = 1-c_n(1-\tau)^n$. Hence

$$d\left(\lambda S_\psi(Tx,q,t), \partial S_\psi(Tx,q,t)\right) \geq \alpha_n(1-\lambda) = c_n'(1-\tau)^n,$$

and (i) follows.

(ii) By (i) we have

$$d(T(y), T(S_\phi(x,p,(1-\epsilon)t)) = \mathrm{d}(T(y), S_\psi(x,q,(1-\epsilon)t)) \geq c_0(1-(1-\epsilon))^n,$$

and hence (ii) follows with $C=C_0$. ☐

2.2. The engulfing property of the sections. The sections of a convex function whose Monge-Ampère measure satisfies (DC) satisfy the following property similar to the one enjoyed by the Euclidean balls. Besides the importance of this property in the study of the linearized Monge-Ampère equation, it also allows us to establish that $\mathbf{R}^n$ equipped with the Monge-Ampère measure μ and the family of sections becomes a space of homogeneous type; see [A-F-T].

Theorem 2.2 (Engulfing property)**.** *Assume that the Monge-Ampère measure μ associated with ϕ satisfies* (DC). *Then there exists a constant $\theta>1$ such that if $y\in S_\phi(x_0,p,t)$, then $S_\phi(x_0,p,t)\subset S_\phi(y,q,\theta t)$ for all $q\in\nabla\phi(y)$.*

Proof. Let T be an affine transformation that normalizes the section $S_\phi(x_0,p,2t)$; that is,

$$B(0,\alpha_n)\subset T(S_\phi(x_0,p,2t))\subset B(0,1).$$

Let $\psi(y)=\phi(T^{-1}y)$, $q_1=(T^{-1})^t p$, and

$$\phi^*(y)=\psi(y)-\psi(Tx_0)-q_1\cdot(y-Tx_0)-2t.$$

By (1-0), $T(S_\phi(x_0,p,2t))=S_\psi(Tx_0,q_1,2t)$. If $q_2\in\nabla\phi^*(Ty)$, then by Lemma 1.1

$$|q_2|\leq\frac{2t}{d(Ty,\partial S_\psi(Tx_0,q_1,2t))}.$$

Since $y\in S_\phi(x_0,p,t)$, $Ty\in S_\psi(Tx_0,q_1,t)$. Thus, by Corollary 2.2(i),

$$|q_2|\leq C_1 t.$$

By taking T^{-1}, the desired inclusion is equivalent to

$$S_\psi(Tx_0,q_1,t)\subset S_\psi(Ty,(T^{-1})^t q,\theta t) \qquad \text{for all } q\in\nabla\phi(y).$$

Let $z\in S_\psi(Tx_0,q_1,t)$. We want to show that

$$\psi(z)<\psi(Ty)+(T^{-1})^t q\cdot(z-Ty)+\theta t \qquad \text{for all } q\in\nabla\phi(y).$$

We have $\nabla\phi^* = \nabla\psi - q_1$, and we observe that $q \in \nabla\phi(y)$ if and only if $(T^{-1})^t q \in \nabla\psi(Ty)$. Hence, if $q \in \nabla\phi(y)$, then $(T^{-1})^t q = q_2 + q_1$ with $q_2 \in \nabla\phi^*(Ty)$.

Therefore

$$\begin{aligned}
&\psi(Ty) + (T^{-1})^t q \cdot (z - Ty) + \theta t \\
&= \psi(Ty) + q_2 \cdot (z - Ty) + q_1 \cdot (z - Ty) + \theta t \\
&\geq \psi(Tx_0) + q_1 \cdot (Ty - Tx_0) + q_2 \cdot (z - Ty) + q_1 \cdot (z - Ty) + \theta t \\
&= \psi(Tx_0) + q_1 \cdot (z - Tx_0) + q_2 \cdot (z - Ty) + \theta t \\
&\geq \psi(z) - t + q_2 \cdot (z - Ty) + \theta t = (*) \qquad \text{since } z \in S_\psi(Tx_0, q_1, t).
\end{aligned}$$

Now $|q_2 \cdot (z - Ty)| \leq C_1 t |z - Ty| \leq 2C_1 t$. Hence $q_2 \cdot (z - Ty) \geq -2C_1 t$, and consequently

$$(*) \geq \psi(z) - t - 2C_1 t + \theta t = \psi(z) + (\theta - (2C_1 + 1))t.$$

The property now follows by picking $\theta \geq 2C_1 + 1$. □

2.3. The size of normalized sections. The following theorem gives a quantitative estimate of the size of normalized sections. It says that if two sections intersect and we normalize the largest of them, then the other one looks like a ball with proportional radius at the scale in which the largest section is normalized. The statement below gives a more precise estimate than the one used in [C-G1]; compare with condition (A) in that paper.

Theorem 2.3. *Assume that the Monge-Ampère measure μ associated with ϕ satisfies* (DC). *There exist positive constants K_1, K_2, K_3 and ϵ such that if $S_\phi(z_0, p_0, r_0)$ and $S_\phi(z_1, p_1, r_1)$ are sections with $r_1 \leq r_0$, $S_\phi(z_0, p_0, r_0) \cap S_\phi(z_1, p_1, r_1) \neq \emptyset$ and T is an affine transformation that normalizes $S_\phi(z_0, p_0, r_0)$, then*

$$B\left(Tz_1, K_2 \frac{r_1}{r_0}\right) \subset T(S_\phi(z_1, p_1, r_1)) \subset B\left(Tz_1, K_1 \left(\frac{r_1}{r_0}\right)^\epsilon\right),$$

and $Tz_1 \in B(0, K_3)$.

Proof. Let $\psi(y) = \dfrac{1}{r_0}\phi(T^{-1}y)$ and set

$$Tz_0 = x_0, \quad Tz_1 = x, \quad p = \frac{1}{r_0}(T^{-1})^t p_0, \quad q = \frac{1}{r_0}(T^{-1})^t p_1, \quad t = \frac{r_1}{r_0}.$$

Hence, by (1-0), we have $T(S_\phi(z_0, p_0, r_0)) = S_\psi(x_0, p, 1)$ and $T(S_\phi(z_1, p_1, r_1)) = S_\psi(x, q, t)$. For the rest of the proof, we shall omit the subscript ψ understanding that the defining function in all sections is ψ. Then the inclusions in the theorem are equivalent to

$$B(x, K_2 t) \subset S(x, q, t) \subset B(x, K_1 t^\epsilon).$$

We begin with the proof of the second inclusion. Since $S(x_0, p, 1)$ is normalized, the center of mass $c(S(x_0, p, 1)) = 0$. By Theorem 2.1, given $0 < \tau < 1$, there exists $0 < \lambda < 1$, $\lambda = \lambda(\tau, n)$ such that

$$\begin{aligned}
S(x, q, \tau) \subset \lambda S(x, q, 1) &= \{x_1 + \lambda(y - x_1) : y \in S(x, q, 1), x_1 = c(S(x, q, 1))\} \\
&= \{(1 - \lambda)x_1 + \lambda y : y \in S(x, q, 1), x_1 = c(S(x, q, 1))\}.
\end{aligned}$$

In the same fashion,

$$\begin{aligned} S(x,q,\tau^2) &\subset \lambda S(x,q,\tau) \\ &= \{(1-\lambda)x_2 + \lambda y : y \in S(x,q,\tau), x_2 = c(S(x,q,\tau))\} \\ &\subset \{(1-\lambda)x_2 + \lambda(1-\lambda)x_1 + \lambda^2 y : y \in S(x,q,1), x_1 \\ &= c(S(x,q,1)), x_2 = c(S(x,q,\tau))\}, \end{aligned}$$

and so

$$S(x,q,\tau^{N+1}) \subset \lambda S(x,q,\tau^N).$$

If we set $x_{i+1} = c(S(x,q,\tau^i))$, $i = 0,1,2,\ldots$, then continuing in this way we obtain

$$S(x,q,\tau^N) \subset \{(1-\lambda)\sum_{i=0}^{N-1} \lambda^i x_{N-i} + \lambda^N y : y \in S(x,q,1)\}.$$

If $x^* \in S(x_0,p,1) \cap S(x,q,t)$, then Theorem 2.2 implies that $S(x,q,1) \subset S(x^*,q',\theta)$ and $S(x_0,p,1) \subset S(x^*,q',\theta)$ for all $q' \in \nabla\psi(x^*)$. Again, by the engulfing property, the last inequality implies that $S(x^*,q',\theta) \subset S(x_0,p,\theta^2)$. On the other hand, by the convexity of ψ,

$$\begin{aligned} S_\psi(x_0,p,r) &\subset x_0 + r(S_\psi(x_0,p,1) - x_0) \\ &= \{x_0 + r(z - x_0) : z \in S_\psi(x_0,p,1)\}, \qquad r > 1. \end{aligned} \tag{2-5}$$

Hence

$$S(x,q,1) \subset S(x_0,p,\theta^2) \subset x_0 + \theta^2(S(x_0,p,1) - x_0).$$

Since $S(x_0,p,1)$ is normalized, it follows that

$$\begin{aligned} x_0 + \theta^2(S(x_0,p,1) - x_0) &= (x_0 - \theta^2 x_0) + \theta^2 S(x_0,p,1) \\ &\subset B(x_0 - \theta^2 x_0, \theta^2) \subset B(0,K), \end{aligned}$$

with $K = 2\theta^2 - 1$. Then $S(x,q,1) \subset B(0,K)$ and consequently $x_{i+1} = c(S(x,q,\tau^i)) \in B(0,K)$, $i = 0,1,\ldots$.

Let $N \geq 0$ be such that $\tau^{N+1} < t \leq \tau^N$. Then

$$\begin{aligned} S(x,q,t) &\subset S(x,q,\tau^N) \\ &\subset \{y_N + \lambda^N y : y \in S(x,q,1)\} \qquad (y_N = (1-\lambda)\sum_{i=0}^{N-1} \lambda^i x_{N-i}) \\ &\subset B(y_N, \lambda^N K). \end{aligned}$$

We have $N + 1 > \dfrac{\log_\lambda t}{\log_\lambda \tau}$, hence $\lambda^N < \lambda^{(\log_\lambda t/\log_\lambda \tau)-1} = \frac{1}{\lambda} t^{\ln\lambda/\ln\tau}$. Since $|y_N| \leq (1-\lambda)K\dfrac{1}{1-\lambda} = K$, we obtain

$$S(x,q,t) \subset B(y_N, \frac{K}{\lambda}t^\epsilon) \subset B(x, 2\frac{K}{\lambda}t^\epsilon),$$

where $\epsilon = \dfrac{\ln\lambda}{\ln\tau}$.

Let us now show the first inclusion. If $y \in S(x_0,p,1) \cap S(x,q,t)$, then by the engulfing property $S(x,q,t) \subset S(y,q',\theta t)$ and $S(x_0,p,1) \subset S(y,q',\theta)$ for all $q' \in$

$\nabla\psi(y)$. Again, by the engulfing property, Theorem 2.2, $S(y,q',\theta) \subset S(x_0,p,\theta^2)$ and consequently $S(x,q,t) \subset S(x_0,p,\theta^2)$, since $t \le 1$. By (2-5),

$$S(x_0,p,3\theta^2) \subset \{x_0 + 3\theta^2(y-x_0) : y \in S(x_0,p,1)\} \subset B(0,K),$$

with $K = 6\theta^2 - 1$.

Let

$$\psi^*(z) = \psi(z) - \psi(x_0) - p\cdot(z-x_0) - 3\theta^2.$$

We claim that

$$(**) \qquad \nabla\psi^*\left(S(x_0,p,2\theta^2)\right) \subset B(0,C\theta^2)$$

with a universal constant C. To show the claim, we first observe that if $\bar{\mu}$ is the Monge-Ampère measure associated with ψ and $S(x_0,p,1)$ is normalized, then by Proposition 1.1 we have $\bar{\mu}(S(x_0,p,1)) \approx 1$. Hence by the doubling property

$$\bar{\mu}(S(x_0,p,2\theta^2)) \approx C(\theta), \qquad \bar{\mu}(S(x_0,p,3\theta^2)) \approx C(\theta).$$

By Aleksandrov's estimate applied to $\psi_2(x) = \psi(x) - \psi(x_0) - p\cdot(x-x_0) - 2\theta^2$ on the section $S(x_0,p,2\theta^2)$, we obtain

$$(\theta^2)^n \le Cd(S(x_0,p,\theta^2), \partial S(x_0,p,2\theta^2))\bar{\mu}(S(x_0,p,2\theta^2)).$$

Thus

$$\mathrm{d}(S(x_0,p,\theta^2), \partial S(x_0,p,2\theta^2)) \ge C.$$

A similar argument yields

$$\mathrm{d}(S(x_0,p,2\theta^2), \partial S(x_0,p,3\theta^2)) \ge C.$$

Hence by applying Lemma 1.1 to the function ψ^* in the set $S(x_0,p,2\theta^2)$, we obtain (**).

Let $x \in S(x_0,p,\theta^2)(\subset B(0,K))$. We shall pick K_2 such that $B(x,K_2t) \subset S(x,q,t)$. Since $\mathrm{d}(S(x_0,p,\theta^2), \partial S(x_0,p,2\theta^2)) \ge C_1$ and $t \le 1$, it follows that $B(x,C_1t/4) \subset S(x_0,p,2\theta^2)$. Let $y \in B(x,K_2t)$ with $K_2 \le C_1/4$. If $q' \in \nabla\psi(y)$, then $\psi(x) \ge \psi(y) + q'\cdot(x-y)$. By (**), $|q'-q| \le 2C\theta^2$ and therefore

$$\begin{aligned} 0 \le \psi(y) - \psi(x) - q\cdot(y-x) &\le -q'\cdot(x-y) - q\cdot(y-x) \\ &\le 2C\theta^2|y-x| \le 2C\theta^2K_2t < t, \end{aligned}$$

by picking K_2 such that $2C\theta^2K_2 < 1$. Thus $y \in S(x,q,t)$. The proof of Theorem 2.3 is now complete. □

Remark 2.3. Theorem 2.3 implies that if the sections of the convex function ϕ are bounded sets and the corresponding Monge-Ampère measure satisfies (DC), then ϕ is strictly convex. In fact, if $P_1 = (x_1,\phi(x_1))$ and $P_2 = (x_2,\phi(x_2))$ are points such that the segment $\overline{P_1P_2}$ is contained in the graph of ϕ and $z_0 = t_0x_1 + (1-t_0)x_2$, $0 < t_0 < 1$, then any supporting hyperplane of ϕ at the point $(z_0,\phi(z_0))$ contains $\overline{P_1P_2}$. Then $\overline{P_1P_2} \subset S_\phi(z_0,p,t)$ for $p \in \nabla\phi(z_0)$ and all $t > 0$. Then by theorem 2.3 the segment $\overline{P_1P_2}$ reduces to a point.

Also as a consequence of Theorem 2.3 we obtain the following important result in the study of the solutions of the linearized Monge-Ampère equation, [C-G2].

Theorem 2.4. *Assume that the Monge-Ampère measure μ associated with ϕ satisfies* (DC)*. Then*

(i) *there exist* $C_0 > 0$ *and* $p_1 \geq 1$ *such that for* $0 < r < s \leq 1$, $t > 0$ *and* $x \in S_\phi(x_0, p, rt)$ *we have*

$$S_\phi(x, q, C_0(s-r)^{p_1} t)) \subset S_\phi(x_0, p, st);$$

(ii) *there exist* $C_1 > 0$ *and* $p_1 \geq 1$ *such that for* $0 < r < s < 1$, $t > 0$ *and* $x \in S_\phi(x_0, p, t) \setminus S_\phi(x_0, p, st)$ *we have*

$$S_\phi(x, q, C_1(s-r)^{p_1} t) \cap S_\phi(x_0, p, rt) = \emptyset.$$

Proof. (i) Let T be an affine transformation normalizing $S_\phi(x_0, p, st)$. Then by (1-0),

$$T\left(S_\phi(x_0, p, st)\right) = S_\psi(Tx_0, q_1, s)$$

where $\psi(y) = \dfrac{1}{t}\phi(T^{-1}y)$, and $q_1 = \dfrac{1}{t}(T^{-1})^t p$. Also

$$T\left(S_\phi(x_0, p, rt)\right) = S_\psi(Tx_0, q_1, r),$$
$$T\left(S_\phi(x, q, C_0(s-r)^{p_1} t)\right) = S_\psi(Tx, q_2, C_0(s-r)^{p_1}),$$

where $q_2 = \dfrac{1}{t}(T^{-1})^t q$. To show (i), it is enough to prove that if $Tx \in S_\psi(Tx_0, q_1, r)$, then

$$S_\psi(Tx, q_2, C_0(s-r)^{p_1}) \subset S_\psi(Tx_0, q_1, s). \tag{2-6}$$

Set $\bar{r} = \dfrac{r}{s} < 1$. Let $\delta < s$ and $z \in S_\psi(Tx, q_2, \delta)$. Then

$$\begin{aligned}\psi(z) &\leq \psi(Tx) + q_2 \cdot (z - Tx) + \delta \\ &\leq \psi(Tx_0) + q_1 \cdot (Tx - Tx_0) + r + q_2 \cdot (z - Tx) + \delta \\ &= \psi(Tx_0) + q_1 \cdot (z - Tx_0) + r + (q_2 - q_1) \cdot (z - Tx) + \delta.\end{aligned}$$

We have $q_1 \in \nabla\psi(Tx_0)$, $q_2 \in \nabla\psi(Tx)$, and $Tx, Tx_0 \in S_\psi(Tx_0, q_1, r)$. By applying Lemma 1.1 to the function $h(x) = \psi(x) - \psi(Tx_0) - q_1 \cdot (x - Tx_0) - s$ on the set $S_\psi(Tx_0, q_1, s)$, and by using Corollary 2.2(i) it follows that

$$\left(\nabla\psi - q_1\right)(Tx) \subset B\left(0, \frac{-h(Tx)}{\mathrm{d}(Tx, \partial S_\psi(Tx_0, q_1, s))}\right) \subset B(0, \frac{C}{(1-\bar{r})^n}).$$

This implies that $|q_2 - q_1| \leq \dfrac{Cs}{(1-\bar{r})^n} = \dfrac{Cs^{n+1}}{(s-r)^n}$. Therefore, by Theorem 2.3,

$$\begin{aligned}\psi(z) &< \psi(Tx_0) + q_1 \cdot (z - Tx_0) + r + (q_2 - q_1) \cdot (z - Tx) + \delta \\ &\leq \psi(Tx_0) + q_1 \cdot (z - Tx_0) + r + \frac{Cs^{n+1}}{(s-r)^n} K_1 \left(\frac{\delta}{s}\right)^\epsilon + \delta.\end{aligned}$$

Thus, if

$$\delta \leq \min\left\{\left(\frac{(s-r)^{n+1}}{2K_1 C}\right)^{1/\epsilon}, \frac{s-r}{2}\right\},$$

then (2-6) follows with $C_0 = \left(\dfrac{1}{2K_1 C}\right)^{1/\epsilon}$ and $p = \dfrac{n+1}{\epsilon}$.

(ii) Let T normalize $S_\phi(x_0, p, 2t)$ and $\psi(y) = \dfrac{1}{2t}\phi(T^{-1}y)$. Then

$$T\left(S_\phi(x_0, p, 2t)\right) = S_\psi(Tx_0, q_1, 1),$$

with $q_1 = \frac{1}{2t}(T^{-1})^t p$. It is sufficient to show that if

$$Tx \in S_\psi(Tx_0, q_1, \frac{1}{2}) \setminus S_\psi(Tx_0, q_1, \frac{s}{2}),$$

then

$$S_\psi(Tx, q_2, C_1 \frac{(s-r)^{p_1}}{2}) \cap S_\psi(Tx_0, q_1, \frac{r}{2}) = \emptyset.$$

We have $q_1 \in \nabla\psi(Tx_0)$ and $q_2 \in \nabla\psi(Tx)$. By Corollary 2.2(i) and Lemma 1.1, $|q_2 - q_1| \le C$. Let $\delta < 1$ and $z \in S_\psi(Tx, q_2, \delta)$. Then by Theorem 2.3,

$$\begin{aligned}
\psi(z) &\ge \psi(Tx) + q_2 \cdot (z - Tx) \\
&\ge \psi(Tx_0) + q_1 \cdot (Tx - Tx_0) + \frac{s}{2} + q_2 \cdot (z - Tx) \\
&= \psi(Tx_0) + q_1 \cdot (z - Tx_0) + \frac{s}{2} + (q_2 - q_1) \cdot (z - Tx) \\
&\ge \psi(Tx_0) + q_1 \cdot (z - Tx_0) + \frac{s}{2} - CK_1\delta^\epsilon \\
&> \psi(Tx_0) + q_1 \cdot (z - Tx_0) + \frac{r}{2},
\end{aligned}$$

if δ satisfies $\delta < \left(\frac{(s-r)}{2CK_1}\right)^{p_1}$, $p_1 = 1/\epsilon$, and the desired conclusion holds with $C_1 \le \frac{1}{(2CK_1)^{p_1}}$. □

From Theorem 2.4, we conclude that there exists $\delta > 0$ such that if $x \in S_\phi(x_0, p, \frac{3}{4}t) \setminus S_\phi(x_0, p, \frac{1}{2}t)$, then

$$S_\phi(x, q, \delta t) \subset S_\phi(x_0, p, t) \setminus S_\phi(x_0, p, \frac{t}{4}).$$

2.4. A result of Caffarelli. In [C1], the section $\Sigma_\phi(x_0^*, p, \delta)$ is defined by

$$\Sigma_\phi(x_0^*, p, \delta) = \{x : \phi(x) - \phi(x_0^*) < p \cdot (x - x_0^*) + \delta\},$$

where $p \in \nabla\phi(\mathbf{R}^n)$ and $\delta > 0$. Note that p may or may not be the slope of a supporting hyperplane of ϕ at $(x_0^*, \phi(x_0^*))$. It was shown in [C1] that given $\delta > 0$ and x_0^* there exists p such that $\Sigma_\phi(x_0^*, p, \delta)$ has x_0^* as its center of mass. From now on, we shall assume that x_0^* is the center of mass of $\Sigma_\phi(x_0^*, p, \delta)$.

The following proposition shows the relationship between the sections S and Σ.

Proposition 2.1. *We have the following.*

(i) *Let x_0^* be the center of mass of $S_\phi(x_0, p, t)$. There exists $\frac{\alpha_n}{2} \le \eta \le 1$ depending on ϕ, x_0 and t such that*

$$S_\phi(x_0, p, t) = \Sigma_\phi(x_0^*, p, \eta t).$$

(ii) *Let $\Sigma_\phi(x_0^*, p, t)$ be a section with x_0^* as its center. Then there exist $x_0 \in \Sigma_\phi(x_0^*, p, t)$ with $p \in \nabla\phi(x_0)$ and $1 \le \gamma \le 2\alpha_n^{-1}$ such that*

$$\Sigma_\phi(x_0^*, p, t) = S_\phi(x_0, p, \gamma t).$$

Proof. We first prove (i). Let T be an affine transformation that normalizes $S_\phi(x_0,p,t)$. We have $Tx_0^*=0$. Let $\psi(y)=\dfrac{\phi(T^{-1}y)-\phi(x_0)-q\cdot(y-Tx_0)}{t}$, $q=(T^{-1})^t p$. It is easy to check that

$$T\Sigma_\phi(x_0^*,p,\eta t)=\Sigma_\psi(0,0,\eta).$$

Hence, to prove (i), it suffices to show that

$$S_\psi(Tx_0,0,1)=\Sigma_\psi(0,0,\eta)\qquad\text{for some }\eta\in\Big[\frac{\alpha_n}{2},1\Big]. \tag{2-7}$$

From the proof of Lemma 2.1, $\eta=1-\psi(0)\in\Big[\dfrac{\alpha_n}{2},1\Big]$. Then writing

$$S_\psi(Tx_0,0,1)=\{y:\psi(y)-\psi(0)<1-\psi(0)\},$$

we have that (2-7) holds with $\eta=1-\psi(0)$.

Condition (ii) can be proved in a similar way. Let x_0^* be the center of mass of $\Sigma_\phi(x_0^*,p,t)$ and let T normalize $\Sigma_\psi(x_0^*,p,t)$. There exists x_0 such that $p\cdot x+\lambda=\ell_{x_0}(x)$ is a supporting hyperplane to ϕ at $(x_0,\phi(x_0))$. Then

$$\Sigma_\phi(x_0^*,p,t)=\{x:(\phi-\ell_{x_0})(x)-(\phi-\ell_{x_0})(x_0^*)<t\}.$$

Therefore, $x_0\in\Sigma_\phi(x_0^*,p,t)$. Obviously

$$T\Sigma_\phi(x_0^*,p,t)=\Sigma_\psi(0,0,1),$$

where $\psi(y)=\dfrac{\phi(T^{-1}y)-\phi(x_0)-q\cdot(y-Tx_0)}{t}$. Clearly

$$\Sigma_\psi(0,0,1)=\{y:\psi(y)<\psi(0)+1\}=S_\psi(Tx_0,0,\gamma),$$

where $\gamma=1+\psi(0)\in[1,2\alpha_n^{-1}]$. By taking T^{-1}, we complete the proof of the proposition. □

As a consequence, we can now prove the strict convexity of ϕ due to Caffarelli [C1].

Corollary 2.2. *Assume that the Monge-Ampère measure μ associated with ϕ satisfies* (DC). *Given $0<\lambda<\lambda'<1$, there exist $0<\delta,\tau<1$ such that if $x\in S_\phi(x_0,p,\lambda)$, then*

(i) *there exists p_1 such that $\Sigma_\phi(x,p_1,\delta)\subset S_\phi(x_0,p,\lambda')$, and*
(ii) *if $q\in\nabla\phi(x)$ and $q'\in\nabla\phi(y)$, then $\langle q'-q,y-x\rangle\ge\tau$ for all $y\in\partial S_\phi(x_0,p,\lambda')$.*

Proof. We first prove (i). By Proposition 2.1 and Theorem 2.2, there exist $x'\in\Sigma_\phi(x,p_1,\delta)$ and $\gamma\in[1,2\alpha_n^{-1}]$ such that

$$\begin{aligned}\Sigma_\phi(x,p_1,\delta)&=S_\phi(x',p_1,\gamma\delta)\\&\subset S_\phi(x,q,\theta\gamma\delta)\\&\subset S_\phi(x,q,\frac{2}{\alpha_n}\theta\delta),\end{aligned}$$

with $q\in\nabla\phi(x)$. Since $x\in S_\phi(x_0,p,\lambda)$, by Theorem 2.4(i)

$$S_\phi(x,q,C_0(\lambda'-\lambda)^p)\subset S_\phi(x_0,p,\lambda').$$

Thus, if $\dfrac{2}{\alpha_n}\theta\delta<C_0(\lambda'-\lambda)^p$, then (i) follows.

To prove (ii), by the convexity we have

$$\phi(y) - \phi(x) - q \cdot (y - x) \leq (q' - q) \cdot (y - x).$$

Since $y \in \partial S_\phi(x_0, p, \lambda')$, by Theorem 2.4(i), $y \notin S_\phi(x, q, C_0(\lambda' - \lambda)^p)$ and the result follows. □

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